UDC 533.69

GENERALIZED SIMILARITY LAWS FOR FLOWS AROUND BODIES IN CONDITIONS OF "LOCALIZABILITY" LAW

PMM Vol. 37, №5, 1973, pp. 857-863

A. I. BUNIMOVICH and A. V. DUBINSKII
(Moscow)

(Received March 27, 1973)

It is shown that generalized similarity laws which interrelate the aerodynamic characteristics (lift and drag coefficients) of a body can be established for the general case of nonaffine-similar bodies to which conditions of the localizability law apply, i.e. when the momentum stream at the body surface depends on local properties of the latter (Newtonian hypersonic gas flow, rarified gas flow, effect of light, etc.).

Methods are derived for the construction of complementary bodies and examples of application of the proposed similarity laws are given. These laws are well known and widely applied in practical problems of flow of perfect gas around bodies at various velocities. They make it possible to determine from known aerodynamic characteristics of a given body its characteristics at various Mach numbers and, in some cases, to determine the lift and drag coefficients of affine-similar bodies [1, 2].

Various theories, in the main based on assumptions that the momentum stream at the body surface primarily depends on the local properties of the latter and on the local angle between the normal to the surface and the direction of flight velocity (the so-called "localizability" law), are successfully used in many areas of flight aerodynamics and dynamics. Specific universal relationships between aerodynamic forces and moments are inherent to flows in conditions of the localizability law [3].

It is shown that conditions of the localizability law make it possible to establish generalized similarity laws which interrelate aerodynamic characteristics of bodies, including nonaffine-similar bodies. A particular case of these laws for Newtonian hypersonic gas flow was considered in [4] on a number of additional assumptions.

1. Statement of problem. Let us consider the flow past a plane profile in conditions of the localizability law. The profile is assumed to be bounded from above by a plane and to be at zero angle of attack (Fig. 1). Any biconvex profile may, evidently, be considered as consisting of two profiles one of which is bounded by a plane from above and the other from below.

When assumptions of the localizability law apply, the lift and drag coefficients are usually expressed by formulas

$$c_y = \frac{1}{\lambda} \int_0^{x_f} f_1(\theta, A_i, B_i)\, dx, \quad c_x = \frac{1}{\lambda} \int_0^{x_f} f_2(\theta, A_i, B_i)\, dx \qquad (1.1)$$

where θ is the angle between the tangent to the profile at a given point and the direction of the freestream, A_i and B_i are constants which define the profile and the freestream, and λ is a characteristic dimension. Integration is carried out from the forward stagnation point to the boundary of the "illuminated", i.e. facing the stream, surface of the profile. If the local coefficient of the force acting on the body can be presented in the form

$$\mathbf{c}_f = \omega(\theta)\,\mathbf{n} + \Omega(\theta)\,\mathbf{v},\ \Omega(\theta) = B_1 \sin\theta, \qquad (1.2)$$
$$\omega(\theta) = A_0 + A_1 \sin\theta + (A_2 - B_1) \times \sin^2\theta$$

formula (1.1) can be written as

$$c_y = \frac{1}{\lambda}\int_0^{x_f} \omega(\theta)\,dx,\ c_x = \frac{1}{\lambda}\int_0^{x_f} [\omega(\theta) + B_1]\operatorname{tg}\theta\,dx \qquad (1.3)$$

Sometimes it is more convenient to write (1.3) in the form

$$c_x^* = \frac{1}{\lambda}\int_0^{x_f} \omega(\theta)\operatorname{tg}\theta\,dx,\quad c_x^* = c_x - \frac{B_1 y_f}{\lambda} \qquad (1.4)$$

$$y_f = \int_0^{x_f} \operatorname{tg}\theta\,dx,\ f_2(\theta, A_i, B_i) = \omega(\theta)\operatorname{tg}\theta$$

Formulas (1.3) cover the most interesting cases of hypersonic gas flow analyzed on assumption of Newton's theory ($A_0 = A_1 = B_1 = 0$) in the classical ($A_2 = 2$) or the modified form ($A_2 = c_{p_0}$ for blunt and $A_2 = c_{p^*} / (\mathbf{vn})_0^2$ for pointed bodies [2], where subscript zero denotes parameters at the leading edge); the free-molecule flows of rarified gas with diffusion-mirror reflection ($A_2 = 2\,(2 - \sigma)$ and $B_1 = 2\sigma_\tau$), hypersonic ($A_0 = A_1 = 0$) or in the Schrello approximation [3, 5] ($A_0 = (2 - \sigma)\,S_\infty^{-2}$, $A_1 = \sigma\,(\pi T_w T_\infty^{-1})^{1/2} S_\infty^{-1}$, where σ and σ_τ are the coefficients of normal and tangential reflections, respectively, T_w and T_∞ are the temperatures of the body and of the freestream, respectively, $S_\infty = v\,(2RT_\infty)^{-1/2}$ and R is the gas constant); the effect of a stream of light on the body ($A_0 = 0, B_1 = 1 - e\,(1 - B)$, $A_1 = {}^2/_3\,[1 - e\,(1 - A)]$, $A_2 = 1 + e\,(1 - A)$, where e, A and B are the coefficients of reflectivity, and of normal and tangential momentum, respectively).

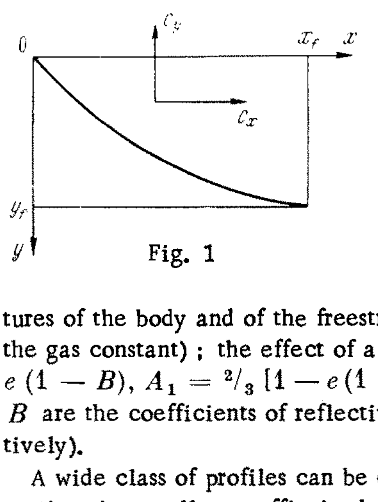

Fig. 1

A wide class of profiles can be derived from the reference profile by certain transformations (generally nonaffine), whose aerodynamic characteristics can be determined from those of the reference profile using the similarity laws which are formulated below.

2. Complementary profiles. Let the shape of the reference profile be defined by equation

$$y^{(0)} = \varphi(x^{(0)}) \qquad (2.1)$$
$$\varphi \geqslant 0,\ \varphi' > 0 \ \text{ for } x^{(0)} \in (0, x_f^{(0)}),\ \varphi(0) = 0$$

where φ is a reasonably smooth function.

We introduce parameter ξ defined by the relationship

$$\xi = \int_0^{x^{(0)}} \Phi[u(t)]\,dt = F(\hat{x}^{(0)}), \quad u(t) = \operatorname{arc\,tg} \varphi'(t) \tag{2.2}$$

where Φ is some reasonably smooth function which in the interval $(0,\pi/2)$ is nonnegative. We can now represent Eq. (2.1) in the parametric form

$$x^{(0)} = F^{-1}(\xi), \quad y^{(0)} = \varphi[F^{-1}(\xi)]$$

Let us construct the complementary profile defined by equations

$$y^{(1)} = y^{(1)}(\xi), \quad x^{(1)} = x^{(1)}(\xi)$$

where $y^{(1)}$ and $x^{(1)}$ are determined by relationships

$$\frac{y^{\cdot(1)}(\xi)}{x^{\cdot(1)}(\xi)} = \operatorname{tg}\Psi(\beta), \qquad x^{\cdot(1)}(\xi) = \frac{1}{\Phi[\Psi(\beta)]} \tag{2.3}$$

$$\beta = \operatorname{arc\,tg}\frac{y^{\cdot(0)}(\xi)}{x^{\cdot(0)}(\xi)}, \qquad x^{\cdot} = \frac{\partial x}{\partial \xi}, \quad y^{\cdot} = \frac{\partial y}{\partial \xi}$$

where Ψ is a certain function.

For any arbitrary ξ the angles of inclination of tangents $\theta^{(0)}(\xi)$ and $\theta^{(1)}(\xi)$ at corresponding points of the reference and the complementary profiles are obviously related by the equation

$$\theta^{(1)} = \Psi(\theta^{(0)}) \tag{2.4}$$

The substitution of $x^{(0)}$ defined by formula (2.2) for parameter ξ in (2.3) yields for the complementary profile equations of the form

$$x^{(1)} = \int_0^{x^{(0)}} Q(u)\,dt, \qquad y^{(1)} = \int_0^{x^{(0)}} Q(u)\operatorname{tg} u\,dt \tag{2.5}$$

$$Q(z) = \Phi(z) \,/\, \Phi[\Psi(z)] \tag{2.6}$$

Thus functions Φ and Ψ transform the reference profile into a complementary one.

The lift and drag of the reference (superscript zero) and the complementary (superscript unity) profiles are defined (within the scope of the localizability law the pattern of flow of these can be different) by

$$c_y^{(0)} = \frac{1}{\lambda^{(0)}}\int_0^{\xi_1}\frac{f_1^{(0)}(\theta)}{\Phi(\theta)}\,d\xi, \quad c_x^{(0)} = \frac{1}{\lambda^{(0)}}\int_0^{\xi_1}\frac{f_2^{(0)}(\theta)}{\Phi(\theta)}\,d\xi \tag{2.7}$$

$$c_y^{(1)} = \frac{1}{\lambda^{(1)}}\int_0^{\xi_1}\frac{f_1^{(1)}[\Psi(\theta)]}{\Phi[\Psi(\theta)]}\,d\xi, \quad c_x^{(1)} = \frac{1}{\lambda^{(1)}}\int_0^{\xi_1}\frac{f_2^{(1)}[\Psi(\theta)]}{\Phi[\Psi(\theta)]}\,d\xi$$

$$\theta \equiv \theta^{(0)}, \quad \xi_1 = F(x_f^{(0)})$$

It is obvious that

$$\int_0^{\xi_1}\frac{d\xi}{\Phi(\theta^{(i)})} \equiv x_f^{(i)}, \qquad \int_0^{\xi_1}\frac{\operatorname{tg}\theta^{(i)}}{\Phi(\theta^{(i)})}\,d\xi = y_f^{(i)}, \quad i = 0, 1 \tag{2.8}$$

Let us assume that there exists some set of constants $a_k^{(0)}$ and $a_k^{(1)}$ in which not all of these are zeros and such that the identity

$$a_1^{(0)}f_1^{(0)}(\theta) + a_2^{(0)}f_2^{(0)}(\theta) + a_3^{(0)}\operatorname{tg}\theta + a_4^{(0)} + Q(\theta)\{a_1^{(1)}f_1^{(1)}[\Psi(\theta)] + a_4^{(1)} + a_2^{(1)}f_2^{(1)}[\Psi(\theta)] + a_3^{(1)}\operatorname{tg}[\Psi(\theta)]\} \equiv 0 \tag{2.9}$$

is satisfied. Then, integrating (2 9) with respect to ξ from zero to ξ_1 with allowance for (1.1) and (2.6) – (2.8), we obtain

$$\sum_{i=0}^{1} \lambda^{(i)} \left[a_1^{(i)}c_y^{(i)} + a_2^{(i)}c_x^{(i)} + a_3^{(i)}\frac{y_f^{(i)}}{\lambda^{(i)}} + a_4^{(i)}\frac{x_f^{(i)}}{\lambda^{(i)}}\right] = 0 \tag{2 10}$$

which defines the relation between the lift and drag coefficients of the reference and the complementary profiles. Formula (2.10) is evidently independent of the shape of the reference profile. Since several linearly-independent sets of $a_k^{(i)}$ are possible, consequently several relationships of the form (2.10) are, also, possible.

The transformation (of the reference profile) is thus determined by functions Ψ and Q of which the latter is not arbitrary, since it must be such that formula (2.6) is satisfied.

If Φ_0 satisfies (2.6), the latter will be also satisfied by function $\Phi(\theta) = \Phi_0 R[\Psi(\theta), \theta]$, where $R(u, v)$ is a symmetric function $(R(u, v) = R(v, u))$. Hence, there exists an infinite set of functions which satisfy (2.6), if the existence of at least one function satisfying (2.6) can be proved. This implies that for one and the same profile transformation parameter ξ may be chosen in various forms.

Some further useful properties of the described transformation should be mentioned. Thus, if for function Q_i we find such $\Phi_i(\theta)$, $i = 1, 2$ that formula (2.6) is satisfied, the corresponding function $\Phi = \Phi_1\Phi_2$ can be found also for $Q = Q_1Q_2$ and for $Q = Q_1^n$ function $\Phi = \Phi_1^n$.

The use of the localizability law is of particular interest in the case of convex profiles. Let us, therefore, establish the conditions under which the transformation does not alter the direction of the profile convexity. Since

$$\frac{d^2y^{(0)}}{dx^{(0)2}} = \varphi''(x^{(0)})$$

$$\frac{d^2y^{(1)}}{dx^{(1)2}} = \frac{\Psi'(\gamma)\varphi''(x^{(0)})}{Q(\gamma)[1+\varphi'^2(x^{(0)})]\cos^2\Psi(\gamma)}, \quad \gamma = u(x^{(0)}) = \operatorname{arc\,tg}\varphi'(x^{(0)})$$

then, evidently, if $\varphi'' \neq 0$ and $\Psi'(\theta) > 0$, the transformation does not affect the direction of convexity, while for $\Psi'(\theta) < 0$ it changes to the opposite direction.

Note that transformation of the reference profile may be repeated several times for various Q and Ψ. Transformations which convert convex-to-concave profile can also be used, since an even number of such transformations successively carried out always convert convex-to-convex profile.

3. Let us consider the transformation of the form

$$x^{(1)} = a_1x^{(0)} + a_2y^{(0)}, \quad y^{(1)} = b_1x^{(0)} + b_2y^{(0)} \tag{3.1}$$

by which the complementary profile is obtained from the reference one by stretching the latter along its axes and turning about the coordinate origin. Let us examine the conditions necessary for the existence of such transformation. Using (2.4), from (3.1) we obtain

$$\int_0^{x^{(0)}} [Q(u) - a_1 - a_2 \operatorname{tg} u]\, dt = 0 \qquad (3.2)$$

$$\int_0^{x^{(0)}} \{Q(u) \operatorname{tg}[\Psi(u)] - b_1 - b_2 \operatorname{tg} u\}\, dt = 0$$

Since (3.2) must be satisfied for any $x^{(0)}$ and $u(t)$, the equalities

$$Q(u) = a_1 + a_2 \operatorname{tg} u, \quad \Psi(u) = \operatorname{arc\,tg} \frac{b_1 + b_2 \operatorname{tg} u}{a_1 + a_2 \operatorname{tg} u} \qquad (3.3)$$

are valid.

Condition (2.8) must be satisfied in addition to conditions (3.3). In this case the former reduces to the requirement of existence of such function Φ that

$$a_1 + a_2 z = R(z) / R\left(\frac{b_1 + b_2 z}{a_1 + a_2 z}\right), \quad z = \operatorname{tg} u, \quad R(z) = \Phi(\operatorname{arc\,tg} z) \qquad (3.4)$$

To establish the existence of the corresponding transformation it is, thus, sufficient to prove that for certain a_1, a_2, b_1 and b_2 there exists a function $R(z)$ which satisfies condition (3.4).

As an example, let us consider the existence of transformation

$$x^{(1)} = a_1 x^{(0)}, \quad y^{(1)} = b_2 y^{(0)}, \quad a_1 > 0, \quad b_2 > 0$$

Condition (3.4) becomes

$$a_1 = R(z) \Big/ R\left(\frac{b_2}{a_1} z\right)$$

It can be shown that the transformation is feasible for any values of parameters a_1 and b_2, except $a_1 = b_2 \neq 1$. The corresponding expressions for function $R(z)$ are as follows: $R(z) \equiv 1$ when $a_1 = 1$; $R(z) = z^c$, where $c = (\log_{a_1} b_2 - 1)^{-1}$ when $a_1 \neq 1$.

4. Examples of application of the proposed method. Let us specify functions Ψ and Q as follows:

$$\Psi(\theta) = {}^1/_2\theta, \quad Q(\theta) = 4\cos^2 {}^1/_2\theta \qquad (4.1)$$

This choice of $Q(\theta)$ is justified, since formula (2.6) is satisfied if we set $\Phi(\theta) = \sin^2\theta$. Transformation (4.1) does not alter the direction of convexity because $\Psi'(\theta) = {}^1/_2 > 0$.

Let the reference and the complementary profiles be subjected to either a Newtonian hypersonic stream or a free molecule stream of rarefied gas. The pattern of flow around these profiles may differ. In this case

$$f_1^{(j)}(\theta, A_i, B_i) = (A_2^{(j)} - B_1^{(j)}) \sin^2\theta, \quad f_2^{(j)}(\theta, A_i, B_i) = f_1^{(j)} \operatorname{tg}\theta, \quad j = 0, 1$$

and formula (2.9) becomes

$$(A_2^{(0)} - B_1^{(0)})(a_1^{(0)} + a_2^{(0)} \operatorname{tg}\theta)\sin^2\theta + a_3^{(0)} \operatorname{tg}\theta + a_4^{(0)} + 4\cos^2 {}^1/_2\theta \times$$
$$\{(A_2^{(1)} - B_2^{(1)})(a_1^{(1)} + a_2^{(1)} \operatorname{tg} {}^1/_2\theta)\sin^2 {}^1/_2\theta + a_3^{(1)} \operatorname{tg} {}^1/_2\theta + a_4^{(1)}\} \equiv 0 \qquad (4.2)$$

After the introduction of the new variable $z = \operatorname{tg} {}^1/_2\theta$ and some elementary transformations, we rewrite (4.2) in the form of conditions for the polynomial in z to be zero. Equating the coefficients of the polynomial to zero, for the determination of $a_i^{(j)}$ we obtain a system of linear equations which has two linearly-independent solutions

$$a_1^{(0)} = B_1^{(0)} - A_2^{(0)}, \quad a_1^{(1)} = A_2^{(1)} - B_1^{(1)}, \quad a_2^{(0)} = a_3^{(0)} = a_4^{(0)} = a_2^{(1)} = a_3^{(1)} = a_4^{(1)} = 0$$

$$a_1^{(0)} = a_4^{(0)} = a_1^{(1)} = a_4^{(1)} = 0, \quad a_2^{(0)} = 2, \quad a_3^{(0)} = 2(B_1^{(0)} - A_2^{(0)}) = -2a_3^{(1)}, \quad a_2^{(1)} = \frac{B_1^{(0)} - A_2^{(0)}}{A_2^{(1)} - B_1^{(1)}}$$

In conformity with equality (2.10) these solutions make it possible to establish the following relationships:

$$\lambda^{(0)}(A_2^{(1)} - B_2^{(1)})c_y^{(0)} = \lambda^{(1)}(A_1^{(0)} - B_1^{(0)})c_y^{(1)} \tag{4.3}$$

$$2\lambda^{(0)}c_x^{(0)} - 2B_1^{(0)}y_f^{(0)} + (A_2^{(0)} - B_1^{(0)})(y_f^{(1)} - 2y_f^{(0)}) - 2\,\frac{B_1^{(0)} - A_2^{(0)}}{B_1^{(1)} - A_2^{(1)}}\,(\lambda^{(1)}c_x^{(1)} - B_1^{(1)}y_f^{(1)}) = 0$$

Solving Eqs.(4.3) for the lift and drag coefficients of the complementary profile we obtain

$$c_y^{(1)} = A\,\frac{\lambda^{(0)}}{\lambda^{(1)}}\,c_y^{(0)}, \quad A = \frac{A_2^{(1)} - B_1^{(1)}}{A_2^{(0)} - B_1^{(0)}} \tag{4.4}$$

$$c_x^{(1)} = \frac{1}{\lambda^{(1)}}\left[A\lambda^{(0)}c_x^{(0)} - AA_2^{(0)}y_f^{(0)} - \frac{1}{2}(3B_1^{(1)} - A_2^{(1)})\right]$$

If the reference profile is subjected to a Newtonian hypersonic stream and the complementary profile to a hypersonic free-molecule stream, it is necessary to set in formulas (4.4) $A_2^{(0)} = 2, B_1^{(0)} = 0, A_2^{(1)} = 2(2 - \sigma)$ and $B_1^{(1)} = 2\sigma_\tau$.

Using formulas (2.4) and (4.1), we write the equation for the complementary profile in the parametric form

$$x^{(1)} = 2t + 2\int_0^t \frac{dt}{[1 + \varphi'^2(t)]^{1/2}}, \quad y^{(1)} = 2\int_0^t \frac{\varphi'(t)\,dt}{[1 + \varphi'^2(t)]^{1/2}}, \quad t = x^{(0)} \tag{4.5}$$

Let us consider some of the reference profile series.

a) Let the reference profile be an arc of circle of radius r with its center at point $(r, 0)$, whose equation is of the form

$$y^{(0)} = [r^2 - (r - x^{(0)})^2]^{1/2} \tag{4.6}$$

Substituting the expression $\varphi(t) = y^{(0)}(x^{(0)})$ from (4.6) into (4.5), we obtain the equation of the complementary profile

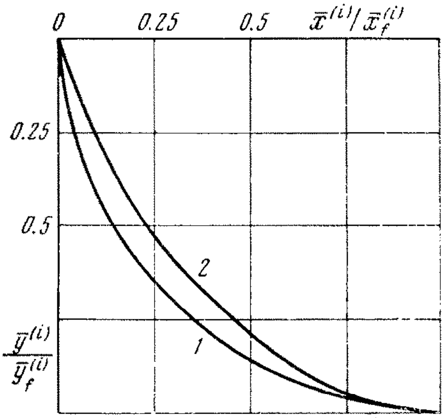

Fig. 2

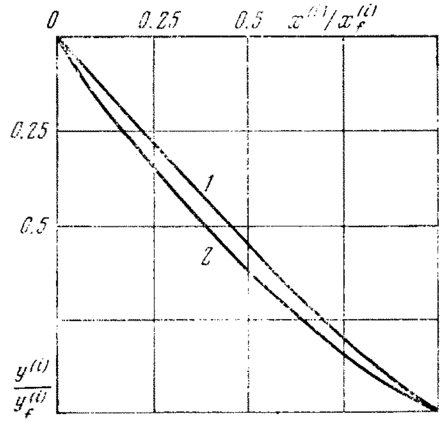

Fig. 3

$$x_+^{(1)} = 2 - (1 + y_+^{(1)})^{1/2} [2 - (y_+^{(1)})^{1/2}] - \arcsin (1 - y_+^{(1)}) + 1/2\,\pi$$

$$x_+^{(i)} = \frac{1}{r} x^{(i)}, \quad y_+^{(i)} = \frac{1}{r} y^{(i)}, \quad i = 0, 1, \quad y_{+_f}^{(1)} = x_{+_f}^{(0)} (2 - x_{+_f}^{(0)})$$

The reference $(x_{+f}^{(0)} = 1)$ and the complementary $(x_{+f}^{(1)} = 3.57)$ profiles are shown in Fig. 2, where they are denoted by 1 and 2, respectively.

b) The reference profile is defined by the equation

$$y^{(0)} = d \ln (x^{(0)} + 1), \quad d > 0$$

From the second formula in (4.3) follows that the equation of the complementary profile is of the form

$$y^{(1)} = 2d \ln \left[1 + \frac{x^{(1)}}{2(1 + \sqrt{1 + d^2})}\right]$$

$$x_f^{(1)} = 2 (x_f^{(0)} + \sqrt{(x_f^{(0)} + 1)^2 + d^2} - \sqrt{1 - d^2})$$

The reference $(d = 1, x_f^{(0)} = 1)$ and the complementary $(x_f^{(1)} = 3.66)$ profiles, denoted respectively by 1 and 2, are shown in Fig. 3.

REFERENCES

1. Kochin, N. E., Kibel', I. A. and Roze, N. V., Theoretical Hydrodynamics, pt 2. Fizmatgiz, 1963.
2. Chernyi, G. G., Flow of Gas at High Supersonic Velocities. Fizmatgiz, Moscow, 1959.
3. Bunimovich, A. I., Relationship between forces acting on a body moving in a rarified gas, in a stream of light, and in a Newtonian hypersonic stream. Izv. Akad. Nauk SSSR, MZhG, №4, 1973.
4. Jaslow, H., Nonaffine similarity laws inherent in Newtonian impact theory. AIAA Journal, Vol. 8, №11, 1970.
5. Schrello, D. M., Approximate free-molecule aerodynamic characteristics. ARS Journal, Vol. 30, №8, 1960.

Translated by J. J. D.